FORMATION OF AN EDDY CURRENT IN THE THERMAL ACCELERATION OF A COMPLETELY IONIZED PLASMA

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The steady-state plane slowly varying flow of a completely ionized nonviscous quasi-neutral plasma in a shaped channel with continuous metal walls is considered. The Hall effect is taken into account. It is shown that for $\beta \gg 1$, where β is the plasma parameter ($\beta = 8\pi p/B^2$, p is the gas-kinetic pressure of the plasma, and B is the magnetic field strength), the acceleration of the plasma is necessarily accompanied by the appearance of natural electromagnetic fields and an electric current, the distribution of which for small discharge voltages has an eddy-current form. The eddy currents disappear when the discharge voltage is increased. The acceleration of a plasma with isothermal electrons is investigated in detail.

The steady-state flows of a plasma with its own magnetic field in shaped planes and axisymmetric channels for large values of the parameter β have been considered repeatedly (see, for example, [1, 2]). It has been shown that these flows may be accompanied by eddy currents close to the electrodes, and that the eddy currents disappear when the parameter β is reduced. The purpose of this paper is to take the Hall effect into account, i.e., to take into account the effect of the elementary plasma acceleration mechanisms [3] on the formation of the electromagnetic field and currents in an accelerated plasma.

1. In order to simplify the calculations we will consider a stationary plane MHD flow (in the xy plane) of a completely ionized nonviscous plasma in a channel with continuous impenetrable metal walls (the electrodes) (Fig. 1). The magnetic field, due to the flow of the discharge current, is directed along the z axis, in the direction of which the channel is assumed to be infinitely wide.

We will assume that the flow is a slowly varying one, and the magnetic Reynolds number is large ($R_m \gg 1$). Instead of the variable y we will introduce the normalized flow function

$$\rho \mathbf{v} = m^{\cdot} \nabla\psi \times \mathbf{n}_z \tag{1.1}$$

where ρ is the plasma density, v is the plasma (ion) velocity, and m is the mass flow rate per second, i.e., the mass of plasma which passes per second through a transverse cross section of the channel.

In the (x, ψ) variables the system of equations which describes the flow takes the form [4]

$$\begin{gathered}\rho v \frac{\partial v}{\partial x} = -\frac{dP}{dx}, \qquad P(x) = p(\rho) + \frac{B^2}{8\pi} \\ \frac{\nu_m}{c}\frac{\rho v}{m^{\cdot}}\frac{\partial B}{\partial \psi} = -\frac{\partial \varphi_T}{\partial x} + \frac{M}{e\rho}\frac{dP}{dx} \qquad \left(\nu_m = \frac{c^2}{4\pi\sigma}\right) \\ \frac{B}{\rho c} = -\frac{1}{m^{\cdot}}\frac{\partial \varphi_T}{\partial \psi}, \qquad \varphi_T = \varphi + \frac{M}{e}\int \frac{dp_i(\rho)}{\rho}\end{gathered} \tag{1.2}$$

Here φ is the electric potential of the plasma, and σ is the plasma conductivity.

We will assume that the state of the plasma components can be described by the polytropic relations $p_{i,e} = p_{i,e}(\rho)$; the latter hold for $\beta \gg 1$, since the Joule heat emission is small, and the effect of heat transfer or heat supply and radiation can be assumed to be appropriate by suitable choice of the polytropic in-

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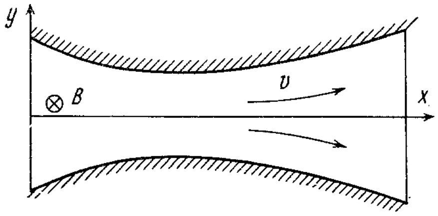

Fig. 1

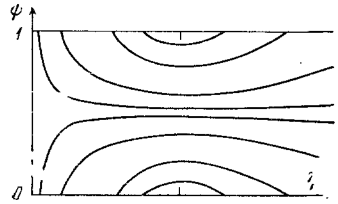

Fig. 2

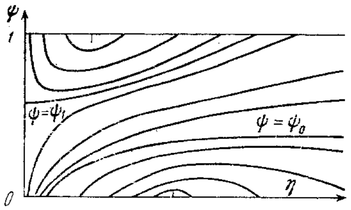

Fig. 3

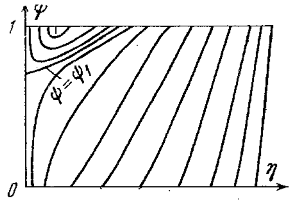

Fig. 4

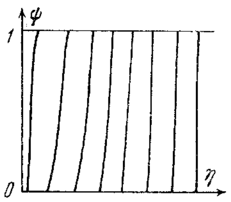

Fig. 5

dex. If $\beta \gg 1$, system (1.2) can be simplified. In fact, in the zeroth approximation with respect to β^{-1} we have $P(x) = p(\rho)$, i.e., $\rho = \rho(x)$, in which case we obtain from the first of Eqs. (1.2)

$$\tfrac{1}{2} v^2 + w(\rho) = F(\psi) \qquad \left(w = \int dp(\rho)/\rho\right) \tag{1.3}$$

If $F(\psi) = \text{const}$, it follows from (1.3) that $v = v(x)$. Since $\sigma \sim T^{3/2}$ (T is the temperature) and, consequently, $\sigma = \sigma(x)$, we can introduce the variable η in place of the variable x:

$$\eta = \int\limits_{x_0}^{x} \frac{\nu_m \rho^2 v}{m^2}\, dx \tag{1.4}$$

where the coordinate x_0 corresponds to the input to the channel. We then obtain the following equations for the potential φ and the magnetic field B:

$$\frac{\partial^2 \varphi}{\partial \psi^2} = \frac{\partial \varphi}{\partial \eta} - \frac{M}{e}\frac{dw_e}{d\eta} \qquad \left(w_e = \int \frac{dp_e(\rho)}{\rho}\right) \tag{1.5}$$

$$B = -\frac{\rho c}{m}\frac{\partial \varphi}{\partial \psi} \tag{1.6}$$

The last terms on the right sides of the third equation of (1.2) and Eq. (1.5) take into account the Hall effect; for $\beta \gg 1$ this is equivalent to taking the term $\nabla p_e / en$ into account in Ohm's law.

It follows from Eq. (1.3) that the acceleration of the plasma with $\beta \gg 1$ has a gas-dynamic form, and the field and current distributions are completely determined by the gas-dynamic nature of the flow. It follows from the presence of the right side in Eq. (1.5) that the plasma potential is not constant even when there is no potential difference between the electrodes, and it follows from Eq. (1.6) that in a plasma there is an inherent (nonconstant) magnetic field, i.e., an electric current flows in the plasma. This is a unique consequence of the transformation of thermal energy of the electrons into energy of directed motion of the plasma (ions), and this transformation must be accompanied by the appearance of an ion-accelerating longitudinal electric field [5] and an "electron wind", i.e., a longitudinal current [3]. When there is no potential difference between the electrodes, there is no discharge current; consequently, the electric distribution in the channel has an eddy-current form.

2. If there is no longitudinal electric current at the input of the channel, and the electrodes are equipotentials, the boundary conditions for Eq. (1.5) are as follows:

$$\varphi(0, \psi) = U\psi, \quad \varphi(\eta, 0) = 0, \quad \varphi(\eta, 1) = U \tag{2.1}$$

Here $U = \text{const}$ is the discharge voltage, and it is assumed that at the cathode $\psi = 0$, and at the anode $\psi = 1$. The solution of Eq. (1.5) with the boundary conditions (2.1) has the form

$$\varphi = U\psi - 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi(2k+1)} \cos\left[\pi(2k+1)\left(\psi - \frac{1}{2}\right)\right] \int_0^{\eta} \frac{dw_e(\tau)}{d\tau} \exp\left[-\pi^2(2k+1)^2(\eta-\tau)\right] d\tau \tag{2.2}$$

The evenness of the last term on the right side of (2.2) in ψ with respect to the line $\psi = 1/2$ is a fairly obvious fact, which follows from the symmetry of the problem for $U = 0$.

As a concrete example we will consider the flow of a plasma with isothermal electrons (T_e = const = T) and ions ($T_i = T_e$).

In this case $w_e = w/2$, and we have from Eq. (1.3)

$$\frac{v^2}{2} + \frac{2T}{M} \ln \frac{\rho}{\rho(0)} = \frac{v^2(0)}{2} \tag{2.3}$$

If we choose the dependence $\rho(\eta)$ in the form

$$\rho = \rho(0) \exp[-\eta] \tag{2.4}$$

we find from (2.3)

$$v = [v^2(0) + 4T\eta / M]^{1/2} \tag{2.5}$$

Relation (2.2) gives

$$\varphi = U\psi + \frac{T}{2e}\left[\left(\psi - \frac{1}{2}\right)^2 - \frac{1}{4}\right] + \frac{4T}{e} \sum_{k=0}^{\infty} (-1)^k \exp\left[-\pi^2(2k+1)^2\eta\right] \frac{\cos[\pi(2k+1)(\psi - 1/2)]}{\pi^3(2k+1)^3} \tag{2.6}$$

Consequently,

$$B = -\frac{\rho(0)c}{m^{\cdot}} \exp[-\eta] \left\{U + \frac{T}{e}\left(\psi - \frac{1}{2}\right) - 4\frac{T}{e} \sum_{k=0}^{\infty} (-1)^k \frac{\sin[\pi(2k+1)(\psi - 1/2)]}{\pi^2(2k+1)^2} \exp\left[-\pi^2(2k+1)^2\eta\right]\right\} \tag{2.7}$$

For $\eta = 0$ the two last terms in the curly brackets cancel one another out. Assuming an exponential fall in the terms of the series, we can write approximately

$$B \approx -\frac{\rho(0)c}{m^{\cdot}} \exp(-\eta) \left\{U + \frac{T}{e}\left(\psi - \frac{1}{2}\right)[1 - \exp(-\pi^2\eta)]\right\} \tag{2.8}$$

Introducing the dimensionless parameter $\varkappa$

$$\varkappa = 2eU / T \geqslant 0$$

we can write expression (2.8) in the form

$$B \approx B(0) \exp(-\eta) \{1 + \varkappa^{-1}(2\psi - 1)[1 - \exp(-\pi^2\eta)]\} \tag{2.9}$$

where $B(0) = -\rho(0)cU/m^{\cdot}$ is the magnetic field at the input to the channel ($\eta = 0$).

We will consider expression (2.9) in more detail. The function $B(\eta, \psi)$ decreases monotonically as ψ increases ($B(0) \leq 0$), reaching a maximum B^* and a minimum B_* at the electrodes $\psi = 0$ and $\psi = 1$. In this case

$$\begin{aligned} B^*(\eta) &= B(\eta, 0) = B(0) \exp(-\eta) \{1 - \varkappa^{-1}[1 - \exp(-\pi^2\eta)]\} \\ B_*(\eta) &= B(\eta, 1) = B(0) \exp(-\eta) \{1 + \varkappa^{-1}[1 - \exp(-\pi^2\eta)]\} \end{aligned} \tag{2.10}$$

The functions $B^*(\eta)$ and $B_*(\eta)$ have extremal points η_c, which correspond to the centers of the eddy structure of the electric current. In this case

$$\eta_c|_{\psi=0} = \pi^{-2} \ln\left|\frac{1+\pi^2}{1-\varkappa}\right|, \qquad \eta_c|_{\psi=1} = \pi^{-2} \ln\left|\frac{1+\pi^2}{1+\varkappa}\right| \tag{2.11}$$

The equation of the line $\psi = \psi_0(\eta)$, which corresponds to zero magnetic field, has the form

$$\psi_0(\eta) = \frac{1}{2} - \frac{\varkappa}{2} [1 - \exp(-\pi^2\eta)]^{-1} \tag{2.12}$$

Hence, the line $B = 0$ exists for $\varkappa \leq 1$. If $\psi > \psi_0$, then $B < 0$, and if $\psi < \psi_0$, then $B > 0$. For $\varkappa = 0$ $\psi_0(\eta) \equiv 1/2$, and for $\varkappa > 0$ and $\eta \to \infty$ $\psi_0 \to (1 - \varkappa)/2$.

We will first consider the case $\varkappa = 0$, when $B(0) = 0$. In this case $\eta_c|_{\psi=0} = \eta_c|_{\psi=1} = \pi^{-2} \ln(1 + \pi^2)$, and the extremal points of the lines of electric current $B(\eta, \psi)$ = const coincide with η_c.

The current distribution pattern in the channel for $\varkappa = 0$ is shown in Fig. 2. When the parameter $\varkappa$ is increased the magnetic field at the input to the channel differs from zero (negatively) and there are two separatrices which limit the eddy structure region, one of them being the line $\psi_0(\eta)$ of zero magnetic field, and the other the line $B(\eta, \psi) = B(0)$, i.e.,

$$\psi = \psi_1(\eta) = \frac{1}{2} + \frac{\varkappa}{2} \frac{\exp(\eta) - 1}{1 - \exp(-\pi^2 \eta)} \tag{2.13}$$

The separatrix $\psi = \psi_1(\eta)$ has a positive derivative $d\psi_1/d\eta$, and it exists (and intersects the anode, i.e., reaches a value $\psi = 1$) for $\varkappa \leq \pi^2$. It follows from (2.11) that when $\varkappa$ increases the eddy current close to the cathode shifts towards larger η, and the eddy current close to the anode is shifted towards the channel input. The electric current distribution pattern for $0 < \varkappa < 1$ is shown in Fig. 3. In the case when $\varkappa \geq 1$ the eddy currents close to the cathode and the separatrix $\psi = \psi_0(\eta)$ disappear, and $B < 0$ over the whole channel (Fig. 4). When the parameter $\varkappa$ reaches a value of π^2 the eddy current near the anode also disappears (the separatrix $\psi_1(\eta)$ contracts to the point $\eta = 0$, $\psi = 1$ at the input of the channel), and for $\varkappa > \pi^2$ the current distribution in the channel does not have an eddy structure (Fig. 5).

The condition $\beta \gg 1$, i.e., $B^2/8\pi p \ll 1$ can be written in the following form:

$$\frac{\rho(0) c^2 MT (1+\varkappa)^2}{32 \pi e^2 m^{\cdot 2}} \ll 1 \tag{2.14}$$

If $v^2(0) \ll 4T/M$, condition (2.14) takes the form

$$\frac{\mathrm{e}}{16} \frac{c^2 (1+\varkappa)^2}{\omega_p^2 f^{*2}} \ll 1 \qquad (\mathrm{e} = 2.718\ldots) \tag{2.15}$$

where $\omega_p^2 = 4\pi e^2 n(0)/M$ and f^* is the width of the channel at the critical cross section (i.e., where $df/dx = 0$).

If $v^2(0) \gg 4T/M$, we obtain from (2.14)

$$\frac{T}{Mv^2(0)} \frac{c^2 (1+\varkappa)^2}{8\omega_p^2 f^2(0)} \ll 1 \tag{2.16}$$

It follows from conditions (2.15) and (2.16) that the discharge voltage must not be too high. An increase in the parameter $\varkappa$ corresponds to a reduction in β. Hence, the eddy currents near the electrodes disappear when β decreases. The exchange parameter $\xi = Mc|B(0)|/4\pi e m^{\cdot}$ in this problem is

$$\xi = \begin{cases} \dfrac{c^2 \varkappa \mathrm{e}}{4\omega_p^2 f^{*2}}, & v^2(0) \ll \dfrac{4T}{M} \\ \dfrac{T}{Mv^2(0)} \dfrac{c^2 \varkappa}{2\omega_p^2 f^2(0)}, & v^2(0) \gg \dfrac{4T}{M} \end{cases} \tag{2.17}$$

Comparing (2.17) with (2.14) and (2.15) we see that

$$\xi \sim \frac{1}{\beta^*} \frac{\varkappa}{(1+\varkappa)^2} \ll 1 \tag{2.18}$$

Here β^* is the maximum value of β. For a given β the exchange parameter ξ is a maximum for $\varkappa = 1$, when the eddy currents close to the cathode disappear.

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